A new approach to the Pontryagin maximum principle for nonlinear fractional optimal control problems

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Abstract

In this paper, we discuss a new general formulation of fractional optimal control problems whose performance index is in the fractional integral form and the dynamics are given by a set of fractional differential equations in the Caputo sense. We use a new approach to prove necessary conditions of optimality in the form of Pontryagin maximum principle for fractional nonlinear optimal control problems. Moreover, a new method based on a generalization of the Mittag-Leffler function is used to solving this class of fractional optimal control problems. A simple example is provided to illustrate the effectiveness of our main result.

Keywords: Fractional calculus, Caputo fractional derivative, Generalized Taylor's formula, Fractional mean value, fractional optimal control, Mittag-Leffler function.

1. Introduction

Fractional optimal control problems (FOCPs) can be regarded as a generalization of classic optimal control problems for which the dynamics of the control system are described by fractional differential equations (FDEs) and might involve a performance index given by fractional integration operator. The reason to formulate and solve FOCPs relies in the fact that there are a significant number of instances in which FDEs describe the behavior of the

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control systems of interest more accurately than the more common integer differential equations. This is the case, for instance, in diffusion processes, control processing, signal processing, stochastic systems, etc. [1].

Fractional calculus (FC) is a field of Mathematics that deals with integrals and derivatives whose order may be an arbitrary real or complex number, thus generalizing the integer-order differentiation and integration. It started more than 300 years ago when the notation for differentiation of non-integer order $1/2$ was discussed between Leibnitz and L'Hospital. Since then, fractional calculus has been developed gradually, being now a very active research area of Mathematical Analysis as attested by the vast number of publications (see [2, 3, 4, 5]). There are several different ways of defining fractional derivatives, and, consequently, different types of FOCPs. However, the ones in the sense the Riemann-Liouville and the Caputo have been more widely used. In most of the works that have been published on FOCPs, the state variable is obtained by the Riemann-Liouville or the Caputo fractional integration of the dynamics, but so far, only integer order integral performance indexes have been considered. It also should be noted that several specific numerical techniques have been developed to solve FOCPs. For more details, see [6, 7, 8, 9].

In this paper, we consider FOCPs for which the performance index is given by an integral of fractional order, and the dynamics are mappings specifying the Caputo fractional derivative of the state variable with respect to time. We use Caputo fractional derivatives because it is the most popular one among physicists and scientists. The reason for this is that fractional derivative of constants are zero. Moreover, the assumptions that we impose on the data of the problem enables a novel approach to the proof based on a generalization of Taylor's expansions and a fractional mean value theorem. Another contribution of the paper consists on an analytic method to solve the fractional differential equation of the illustrative example based on a generalization of the Mittag-Leffler function and α exponential function.

This paper is organized as follows. In the next Section, we present a brief review of fractional integrals and fractional derivatives concept and some basic notions specifically pertinent to this work. In Section 3, we state, discuss and prove necessary conditions of optimality in the form of a Pontryagin Maximum Principle for nonlinear fractional optimal control problems. In Section 4, a simple illustrative example of a FOCP solved by a method based on the Mittag-Leffler function is presented. Finally, in Section 5 we present

some conclusions of this research as well as some open challenges.

2. Some preliminaries in fractional calculus

There are several definitions of a fractional derivative. In this section, we present a review of some definitions and preliminary facts which are particularly relevant for the results of this article [10, 11, 12].

Definition 2.1. Let $f(\cdot)$ be a locally integrable function in interval $[a,b]$. For $t \in [a,b]$ and $\alpha > 0$, the left and right Riemann-Liouville fractional integrals are, respectively, defined by

$$ {}_aI_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau, $$

and

$$ {}_tI_b^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\tau-t)^{\alpha-1} f(\tau) d\tau, $$

where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 2.2. Let $f(\cdot)$ be an absolutely continuous function in the interval $[a,b]$. For $t \in [a,b]$ and $\alpha > 0$, the left and right Riemann-Liouville fractional derivatives are, respectively, defined by

$$ {}_aD_t^{\alpha} f(t) = \frac{d^n}{dt^n} \left({}_aI_t^{n-\alpha} f(t) \right) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt} \right)^n \int_a^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau, $$

and

$$ {}_tD_b^{\alpha} f(t) = (-\frac{d}{dt})^n \left({}_tI_b^{n-\alpha} f(t) \right) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dt} \right)^n \int_t^b (\tau-t)^{n-\alpha-1} f(\tau) d\tau, $$

where $n \in \mathbb{N}$ is such that $n-1 < \alpha \leq n$, and $\Gamma(\cdot)$ is as in Definition 2.1.

Definition 2.3. Let $f(\cdot)$ be an integrable continuous function in the $[a,b]$. For $t \in [a,b]$ and $\alpha > 0$, the left and the right Caputo fractional derivatives are, respectively, defined by

$$ {}_{a}^{C}D_t^{\alpha} f(t) = {}_{a}I_t^{n-\alpha} \frac{d^n}{dt^n} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, $$

and

$$ {}_{t}^{C}D_b^{\alpha} f(t) = {}_{t}I_b^{n-\alpha} \left(-\frac{d}{dt}\right)^n f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b (\tau-t)^{n-\alpha-1} f^{(n)}(\tau) d\tau, $$

where $n \in \mathbb{N}$ is such that $n-1 < \alpha \leq n$.

Remark 2.1. If $\alpha = n \in \mathbb{N}_0$, then the Caputo and Riemann-Liouville fractional derivative coincides the ordinary derivative $\dfrac{d^n f(t)}{dt^n}$.

Remark 2.2. The Caputo fractional derivative of a constant is always equal to zero. This is not the case with the Riemann-Liouville fractional derivative.

Theorem 2.1 (see [13]). Let $\alpha > 0$ and $f(\cdot)$ be a differentiable function in $[a, b]$, then

$$ {}_{a}^{C}D_t^{\alpha}\, {}_{a}I_t^{\alpha} f(t) = f(t), \quad {}_{t}^{C}D_b^{\alpha}\, {}_{t}I_b^{\alpha} f(t) = f(t), $$
$$ {}_{a}D_t^{\alpha}\, {}_{a}I_t^{\alpha} f(t) = f(t), \quad {}_{t}D_b^{\alpha}\, {}_{t}I_b^{\alpha} f(t) = f(t), $$

and

$$ {}_{a}I_b^{\alpha}\, {}_{a}^{C}D_t^{\alpha} f(t) = f(b) - f(a), \quad {}_{b}I_a^{\alpha}\, {}_{t}^{C}D_b^{\alpha} f(t) = f(a) - f(b). $$

Theorem 2.2. Fractional integration by parts.

Let $0 < \alpha < 1$, $f(\cdot)$ be a differentiable function in interval $[a, b]$ and $g(\cdot) \in L_1([a, b])$. Then the following integration by parts formula holds

$$ \int_a^b g(t)\, {}_{a}^{C}D_t^{\alpha} f(t) dt = \int_a^b f(t)\, {}_{t}D_b^{\alpha} g(t) dt + \left[{}_{t}I_b^{1-\alpha} g(t) f(t)\right]_a^b $$

and

$$ \int_a^b g(t)\, {}_{t}^{C}D_b^{\alpha} f(t) dt = \int_a^b f(t)\, {}_{a}D_t^{\alpha} g(t) dt - \left[{}_{a}I_t^{1-\alpha} g(t) f(t)\right]_a^b. $$

Another important auxiliary result to prove our Maximum Principle is the generalization of the Bellman-Gronwall Lemma for fractional differential systems. Here, we will consider the following integral from extracted from [14].

Theorem 2.3. Generalized Bellman-Gronwall inequality.

Suppose $\alpha > 0$, $t \in [0, T)$ and the functions $a(t)$, $b(t)$and $u(t)$ are a non-negative and continuous functions on $0 \leq t < T$ with

$$u(t) \leq a(t) + b(t) \int_0^t (t-s)^{\alpha-1} u(s) ds,$$

where $b(t)$ is a bounded and monotonic increasing function on $[0, T)$, then

$$u(t) \leq a(t) + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(b(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)} (t-s)^{n\alpha-1} a(s) \right] ds, \quad t \in [0, T)$$

Theorem 2.4. Generalized Taylor's formula (cf. [15, 16]).

Let $0 < \alpha \leq 1$, $n \in \mathbb{N}$, $f(\cdot)$ be a continuous function in $[a, b]$, ${}^{C}D_a^{k\alpha} f(\cdot) \in C[a, b]$ $\forall k = 1, \ldots, n$ and ${}^{C}D_a^{(n+1)\alpha} f(\cdot)$ is continuous on $[a, b]$, then $\forall x \in [a, b]$ the generalized Taylor's formula for Caputo fractional derivatives is defined by

$$f(x) = \sum_{k=0}^{n} \frac{(x-a)^{k\alpha}}{\Gamma(k\alpha+1)} \, {}^{C}D_a^{k\alpha} f(a) + R_n(x, a),$$

where

$$R_n(x, a) = {}^{C}D_a^{(n+1)\alpha} f(\xi) \frac{(x-a)^{(n+1)\alpha}}{\Gamma((n+1)\alpha+1)},$$

being, for each $x \in [a, b]$, $a \leq \xi \leq x$, and denoting the Caputo fractional derivative of order α by ${}^{C}D_a^{\alpha}$.

Notice that, if $\alpha = 1$, the generalized Taylor's formula reduces to the classical Taylor's formula.

Lemma 2.1. (see [17]) Let $f \in C[a, b]$, $\alpha > 0$, then there exists some $\xi \in (a, b)$ such that

$$I_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt = f(\xi) \frac{(x-a)^{\alpha}}{\mathit{\Gamma}(1+\alpha)},$$

where ξ the fractional intermediate value. Remark that there might exist more than one ξ satisfying this property.

Definition 2.4. The two-parameter Mittag-Leffler function defined by the power series in the form:

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma[n\alpha + \beta]},$$

where α, and β are positive parameters. When $\beta = 1$, this function is denoted simply by $E_\alpha(\cdot)$. We observe that $E_{0,1}(z) = 1/(1-z)$, $E_{1,1}(z) = \exp z$, $E_{1,2}(z) = (\exp z - 1)/z$, and $E_{1,0}(z) = z \exp z$.

Let $A \in \mathbb{R}^{n\times n}$, then the generalization of the two-parameter Mittag-Leffler function becomes

$$E_{\alpha,\beta}(At^\alpha) = \sum_{n=0}^{\infty} A^n \frac{t^{n\alpha}}{\Gamma[n\alpha + \beta]},$$

and let us define the α exponential matrix function by using Mittag-Leffler function as follows

$$e_\alpha(A,t) = t^{\alpha-1} E_{\alpha,\alpha}(At^\alpha) = t^{\alpha-1} \sum_{n=0}^{\infty} A^n \frac{t^{n\alpha}}{\Gamma[(n+1)\alpha]} \tag{1}$$

The Mittag-Leffler function has several interesting properties. For details see [18, 19, 20, 21].

3. The FOCP statement and its Maximum Principle

In this section, we discuss the FOCP considered in this article, state the associated necessary conditions of optimality, and present its proof which uses an approach that differs from the ones usually adopted for fractional optimal control problems.

Let us consider the simple general problem as follows

$$\begin{aligned}
(\bar{P})\ \text{Minimize} \quad & {}_{t_0}I^{\alpha}_{t_f} L(t, \bar{x}(t), u(t)) \\
\text{subject to} \quad & {}^{C}_{t_0}D^{\alpha}_{t} \bar{x}(t) = \bar{f}(t, \bar{x}(t), u(t)), \quad [t_0, t_f]\ \mathcal{L} - \text{a.e.} && (2) \\
& \bar{x}(t_0) = \bar{x}_0 \in \mathbb{R}^n && (3) \\
& u(t) \in \mathcal{U} && (4)
\end{aligned}$$

where $\mathcal{U} = \{u : [t_0, t_f] \to \mathbb{R}^m : u(t) \in \Omega(t)\}$, $\Omega : [t_0, t_f] \to \mathbb{R}^m$ is a given set valued mapping, $L : \mathbb{R}^n \to \mathbb{R}$ and $\bar{f} : [t_0, t_f] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ are given

functions defining respectively the running cost (or Lagrangian) functional and the fractional dynamics, ${}_{t_0}I_{t_f}^{\alpha}$ is the Riemann-Liouville fractional integral and ${}_{t_0}^{C}D_t^{\alpha}x$ is the left Caputo fractional derivative of order $\alpha > 0$ of the state variable with respect to time.

It is not hard to see that a simple transformation allows us to convert the problem $(\bar{P})$ into an equivalent one, simply by using this assumption ${}_{t_0}^{C}D_t^{\alpha}y(t) = L(t, \bar{x}(t), u(t))$, supplemented by the initial condition $y(t_0) = 0$. Then, we conclude that problem $(\bar{P})$ is equivalent to the one as follows:

$$\begin{aligned}
(P)\ \text{Minimize}\quad & g(x(t_f)) \\
\text{subject to}\quad & {}_{t_0}^{C}D_t^{\alpha}x(t) = f(t, x(t), u(t)), \quad [t_0, t_f]\ \mathcal{L}-\text{a.e.} && (5)\\
& x(t_0) = x_0 \in \mathbb{R}^n && (6)\\
& u(t) \in \mathcal{U}, && (7)
\end{aligned}$$

where now $g(x(t_f)) = y(t_f)$, the state variable $x = col(y, \bar{x})$, i.e., it includes y as a first component with initial value at 0, and the mapping $f = col(L, \bar{f})$, i.e., it has L as first component.

From now on, we consider this as the basic optimal control problem in normal form. We remark that the above problem statement is the simplest one that can be considered containing all the ingredients required for "bona fide" optimal control problem.

Now, we will state the assumptions under which our result will be proved.

(H1) The function g is C_1 in $\mathbb{R}^n$, i.e., continuously differentiable in its domain.

(H2) The function f is C_1 and Lipschitz continuous with constant K_f in x for all $(t, u) \in \{(t, \Omega(t)) : t \in [t_0, t_f]\}$.

(H3) The function f is continuous in (t, u), for all $x \in \mathbb{R}^n$.

(H4) The set valued map $\Omega : [t_0, t_f] \to \mathbb{R}^m$ is compact valued.

(H5) The set $f(t, x, \Omega(t))$ is bounded by a certain positive constant M for all $(t, x) \in [t_0, t_f] \times \mathbb{R}^n$.

These are, by no means, the weakest hypotheses enabling the proof of the maximum principles for FOCPs. However, these ones are of interest in that it allows the particularly simple proof adopted in this article.

Consider

$$H(t, x, p, u) := p^T f(t, x, u),$$

with $p \in \mathbb{R}^n$, to be the Pontryagin function associated to problem (P).

Theorem 3.1 Let (x^*, u^*) be optimal control process for (P). Then, there exists a function $p : [t_0, t_f] \to \mathbb{R}^n$ satisfying

- the adjoint equation

$$ {}_tD^{\alpha}_{t_f} p^T(t) = p^T(t) D_x f(t, x^*(t), u^*(t)), \tag{8}$$

- and the transversality condition

$$p^T(t_f) = \nabla_x g(x^*(t_f)), \tag{9}$$

where the operator ${}_tD^{\alpha}_{t_f}$ is right Riemann-Liouville fractional derivative, and $u^* : [t_0, t_f] \to \mathbb{R}^m$ is a control strategy such that $u^*(t)$ maximizes $[t_0, t_f]$ $\mathcal{L}$-a.e. the map

$$u \to H(t, x^*(t), p(t), u),$$

on $\Omega(t)$.

Proof of Theorem 3.1

The first key idea is that any perturbation of the optimal control u^* that affects the final value of the state trajectory may increase the cost. Thus, the proof relies on the comparison between the optimal trajectory x^* and trajectories x which are obtained by perturbing the optimal control u^*. Let τ be a Lebesgue point in (t_0, t_f), and $\varepsilon > 0$ sufficiently small so that $\tau - \varepsilon \geq t_0$. By Lebesgue point in the fractional context, which define in the next definition.

Definition 3.1. A Lebesgue point of an integrable function $f : \mathbb{R} \to \mathbb{R}$ is a point $t_0 \in \mathbb{R}$ satisfying

$$\lim_{\epsilon \to 0^+} \frac{1}{2\varepsilon} \, {}_{t_0-\varepsilon}I^{\alpha}_{t_0+\varepsilon} \, |f(t) - f(t_0)| \to 0.$$

It is well known that the subset of Lebesgue points of an integrable function f forms a full Lebesgue measure subset.

Now, let us consider the perturbed control strategy $u_{\tau,\varepsilon}$ defined by

$$u_{\tau,\varepsilon}(t) = \begin{cases} \bar{u} & \text{if } t \in [\tau - \varepsilon, \tau) \\ u^*(t) & \text{if } t \in [t_0, t_f] \setminus [\tau - \varepsilon, \tau) \end{cases} \tag{10}$$

where $\bar{u} \in \Omega(t)$ for all $t \in [\tau - \varepsilon, \tau)$, being τ a Lebesgue point of the reference optimal control strategy. Note that, there is no loss of generality of the choice of τ due to the fact that the set Lebesgue points is of full Lebesgue measure.

Let $x_{\tau,\varepsilon}$ be the trajectory associated with $u_{\tau,\varepsilon}$, and with $x_{\tau,\varepsilon}(t_0) = x_0$. Clearly, by definition of optimality of (x^*, u^*),

$$\left\{ \begin{array}{lll} 0 & \leq & g(x_{\tau,\varepsilon}(t_f)) - g(x^*(t_f)) \\ & = & \nabla_x g(x^*(t_f))[x_{\tau,\varepsilon}(t_f) - x^*(t_f)] + o(\varepsilon) \\ & = & \nabla_x g(x^*(t_f))\Phi_\alpha(t_f, \tau)[x_{\tau,\varepsilon}(\tau) - x^*(\tau)] + o(\varepsilon), \end{array} \right. \tag{11}$$

where $\nabla_x g(\cdot)$ is the gradient of $g(\cdot)$, $o(\varepsilon)$ is some positive number satisfying $\lim\limits_{\varepsilon \to 0} \dfrac{o(\varepsilon)}{\varepsilon} = 0$, $\Phi_\alpha(\cdot, \cdot)$ is the state transition matrix for the linear fractional differential system

$${}_{t_0}^{C}D_t^\alpha \xi(t) = D_x f(t, x^*(t), u^*(t))\xi(t),$$

and $x_{\tau,\varepsilon} : [t_0, t_f] \to \mathbb{R}^n$ is the solution to ${}_{t_0}^{C}D_t^\alpha x_{\tau,\varepsilon}(t) = f(t, x_{\tau,\varepsilon}(t), u_{\tau,\varepsilon}(t))$ with $x_{\tau,\varepsilon}(0) = x_0$.
Observe that $x_{\tau,\varepsilon}(t) = x^*(t)$, for all $t \in [t_0, \tau)$.

For all $t \in [\tau - \varepsilon, \tau)$, it is clear that

$$\begin{array}{lll} |x_{\tau,\varepsilon}(t) - x^*(t)| & \leq & {}_{\tau-\varepsilon}I_\tau^\alpha |f(s, x_{\tau,\varepsilon}(s), \bar{u}) - f(s, x^*(s), u^*(s))| ds \\ & \leq & {}_{\tau-\varepsilon}I_\tau^\alpha K_f |x_{\tau,\varepsilon}(s) - x^*(s)| ds + 2M \dfrac{\varepsilon^\alpha}{\Gamma(\alpha+1)} \\ & \leq & \dfrac{\bar{M}\varepsilon^\alpha}{\Gamma(\alpha+1)}, \end{array}$$

where

$$\bar{M} = 2M \left(1 + K_f \sum_{n=1}^{\infty} \frac{\Gamma(\alpha)^{n-1}}{\Gamma(n\alpha+1)} \varepsilon^{n\alpha} \right).$$

It is not difficult to show that this series converges and thus $\bar{M}$ is some finite positive number. The last inequality was obtained by applying Theorem 2.3.

In order to proceed, we need the following auxiliary result.

Lemma 3.1. Consider the general time interval $[a, b]$ and define the function $F(t, x) = f(t, x, \bar{u})$, where $\bar{u}$ is like in (10). Moreover, consider $\tilde{x}(\cdot)$ and $y(\cdot)$ to be, respectively, solutions to the following fractional differential systems:

- ${}_a^C D_t^\alpha \tilde{x}(t) = F(t, \tilde{x}(t))$ with $\tilde{x}(a) = x_a$, and
- ${}_a^C D_t^\alpha y(t) = D_x F(t, \tilde{x}(t)) y(t)$ with $y(a) = \bar{y}\Gamma(\alpha + 1)$.

Then, for all ν positive and sufficiently small real number, we have that $\tilde{x}_\nu(\cdot)$ solution to the system

$$ {}_a^C D_t^\alpha \tilde{x}_\nu(t) = F(t, \tilde{x}_\nu(t)), \quad \tilde{x}_\nu(a) \in x_a + \nu^\alpha \bar{y} + o(\nu^\alpha) B_1^n(0), $$

satisfies on $[a, b]$,

$$ \tilde{x}_\nu(t) \in \tilde{x}(t) + \frac{\nu^\alpha}{\Gamma(\alpha + 1)} y(t) + o(\nu^\alpha) B_1^n(0). $$

Here, $B_1^n(0)$ denotes the closed unit ball of $\mathbb{R}^n$ centered at 0,

Proof of Lemma 3.1
After using Taylor's series of fractional order as defined in Theorem 2.4. We conclude next inequality

$$ \left| {}_a^C D_t^\alpha \left(\tilde{x}(t) + \frac{\nu^\alpha}{\Gamma(\alpha + 1)} y(t) \right) - F(t, \tilde{x}(t) + \frac{\nu^\alpha}{\Gamma(\alpha + 1)} y(t)) \right| \leq o(\nu^\alpha) $$

and we can be extracted this inequality as the follows

$$ \left| {}_a^C D_t^\alpha \tilde{x}(t) + {}_a^C D_t^\alpha y(t) \frac{\nu^\alpha}{\Gamma(\alpha + 1)} - F(t, \tilde{x}(t)) - \frac{\nu^\alpha}{\Gamma(\alpha + 1)} D_x F(t, \tilde{x}(t)) y(t)) \right| \leq o(\nu^\alpha). $$

Clearly, the first and the third terms cancel each other in the left hand side of the inequality, and, thus, we have

$$ \left| {}_a^C D_t^\alpha y(t) \frac{\nu^\alpha}{\Gamma(\alpha + 1)} - \frac{\nu^\alpha}{\Gamma(\alpha + 1)} D_x F(t, \tilde{x}(t)) y(t)) \right| \leq o(\nu^\alpha). $$

By dividing each side by ν^α, where $\frac{o(\nu^\alpha)}{\nu^\alpha} \to 0$, when $\nu \to 0^+$, we conclude immediately fractional linearized differential system

$$ {}_a^C D_t^\alpha y(t) = D_x F(t, \tilde{x}(t)) y(t), \quad y(a) = \bar{y}\Gamma(\alpha+1). \qquad \square $$

Since, from above, $|x_{\tau,\varepsilon}(\tau) - x^*(\tau)| \leq \bar{M}\dfrac{\varepsilon^\alpha}{\Gamma(\alpha+1)}$ for some finite $\bar{M}$, we may apply Lemma 3.1 to the case of $t \in [\tau, t_f]$.
By putting $a = \tau$, $\tilde{x} = x^*$, $y = \xi$, $\nu = \varepsilon$, $\tilde{x} = x_{\tau,\varepsilon}$ and

$$ \bar{y} = f(\tau, x_{\tau,\varepsilon}(\tau), \bar{u}) - f(\tau, x^*(\tau), u^*(\tau)), $$

Lemma 3.2 readily yields, for almost all $t \in [\tau, t_f]$,

$$ x_{\tau,\varepsilon}(t) \in x^*(t) + \frac{\varepsilon^\alpha}{\Gamma(\alpha+1)}\xi(t) + o(\varepsilon^\alpha) B_1^n(0), \tag{12} $$

where $\xi(\cdot)$ satisfies the fractional linearized differential system

$$ \begin{cases} {}_\tau^C D_t^\alpha \xi(t) &= D_x f(t, x^*(t), u^*(t))\xi(t), \qquad \mathcal{L} - \text{a.e. } t \in [\tau, t_f], \\ \xi(\tau) &= f(\tau, x_{\tau,\varepsilon}(\tau), \bar{u}) - f(\tau, x^*(\tau), u^*(\tau)). \end{cases} \tag{13} $$

By putting together (12), and the chain of inequalities in (11) we can immediately write the inequality

$$ \begin{aligned} 0 &\leq \nabla_x g(x^*(t_f))\Phi_\alpha(t_f, \tau)[x_{\tau,\varepsilon}(\tau) - x^*(\tau)] + o(\varepsilon) \\ &\leq \frac{\varepsilon^\alpha}{\Gamma(\alpha+1)} \nabla_x g(x^*(t_f))\Phi_\alpha(t_f, \tau)\xi(\tau). \end{aligned} \tag{14} $$

By putting $p^T(t_f) = -\nabla_x g(x^*(t_f))$ and $p^T(t) = p^T(t_f)\Phi_\alpha(t_f, t)$, we conclude immediately that the adjoint variable $p : [t_0, t_f] \to \mathbb{R}^n$ satisfies the adjoint equation and the transversatility condition, respectively, (8) and (9).
This, together with the definition of $\xi(\tau)$ and the definition of the Pontryagin function, we conclude, after dividing both sides of the inequality above by $\dfrac{\varepsilon^\alpha}{\Gamma(\alpha+1)}$, considering the arbitrariness of $\bar{u} \in \Omega(t)$ and taking the limit as $\varepsilon \to 0^+$ at time τ, that

$$ H(\tau, x^*(\tau), p(\tau), u^*(\tau)) \geq H(\tau, x^*(\tau), p(\tau), \bar{u}). $$

The fact that τ is an arbitrary Lebesgue point in $[t_0, t_f]$ implies that the maximum condition of our main result holds, that is, $u^*(t)$ maximizes, on $\Omega(t)$, the map $u \to H(t, x^*(t), p(t), u)$, $[t_0, t_f]$ $\mathcal{L}$-a.e..

Our main result is proved.

4. Illustrative example

The Pontryagin maximum principle proved in the previous section is now apply to solve a simple problem of resources management that involves minimizing a certain fractional integral subject to given controlled FDEs.

We consider the following problem

$$\text{Minimize} \quad J(u) \tag{15}$$
$$\text{subject to} \quad {}_0^C D_t^\alpha x(t) = u(t)x(t), \quad t \in [0,T], \tag{16}$$
$$x(0) = x_0, \tag{17}$$
$$u(t) \in [0,1], \tag{18}$$

where $J(u) = -{}_0I_T^\alpha (1-u(t))x(t)$, with $0 < \alpha < 1$ and $T > \Gamma(\alpha+1)^{\alpha^{-1}}$. Here ${}_0I_T^\alpha$ is fractional integral and ${}_0^C D_t^\alpha$ is left Caputo fractional derivative.
The variable x represents a natural resource that takes positive values (note that $x_0 > 0$ necessarily) "grows" according to the law (16), where the function u, designated by control, represents the fraction of the available resource that is used to promote further growth. The overall goal is to find the control strategy that maximizes the amount of accumulated resource over the time interval $[0,T]$ given by the fractional integral (15).

First, we consider an additional state variable component y, satisfying

$$ {}_0^C D_t^\alpha y(t) = (1-u(t))x(t), \quad y(0) = 0,$$

in order obtain the problem statement in the form considered in our main result, that is,

$$\begin{array}{rll} \text{Minimize} & -y(T) & \\ \text{subject to} & {}_0^C D_t^\alpha x(t) = u(t)x(t), & x(0) = x_0, \\ & {}_0^C D_t^\alpha y(t) = (1-u(t))x(t), & y(0) = 0, \\ & u(t) \in [0,1]. & \end{array}$$

From Theorem 3.1, the adjoint equation (8) and the transversality condition (9) for this problem are

$$ {}_tD_T^\alpha p_1(t) = [p_1 u^*(t) + p_2(1-u^*(t))], \qquad p_1(T) = 0, \tag{19}$$
$$ {}_tD_T^\alpha p_2(t) = 0, \qquad p_2(T) = 1 \tag{20}$$

where ${}_tD_T^\alpha$ is right Riemann-Liouville fractional derivative of order α. Thus, we have that $p_2(t) \equiv p_2(T) = 1$, and equation (19) becomes

$$ {}_tD_T^\alpha p_1(t) = [(p_1(t) - 1)u^*(t) + 1]. \tag{21}$$

From the maximum condition, we know that $u^*(t)$ maximizes, $\mathcal{L}$-a.e. in $[0, 1]$, the mapping

$$v \to p^T(t) f(t, x^*(t), y^*(t), v) = [p_1(t)v + p_2(t)(1 - v)]x^*(t).$$

Since $p_2 = 1$ and $x^*(t) > 0$ for all $t \in [0, T]$ (this is to conclude from the fact that $x_0 > 0$), the mapping to be maximized can be simplified to $v \to (p_1(t) - 1)v$. Thus, given that the system is time invariant, we have that

$$u^*(t) = \begin{cases} 1 & \text{if } p_1(t) > 1 \\ 0 & \text{if } p_1(t) < 1. \end{cases}$$

Since $p_1(T) = 0$, and $p_1(\cdot)$ is continuous, $\exists b > 0$ s.t. $u^*(t) = 0\ \forall t \in [T - b, T]$. Thus, from (19) we have ${}_tD_T^\alpha p_1(t) = 1$ and, by backwards integration we obtain

$$p_1(t) = \frac{(T - t)^\alpha}{\Gamma(\alpha + 1)} \tag{22}$$

Obviously that, for $t^* = T - (\Gamma(\alpha + 1))^{\frac{1}{\alpha}}$, we obtain $p_1(t^*) = 1$. Now, Let us determine the optimal control for $t < t^*$. Since, independently of the control $p_1(\cdot)$ remains monotonically decreasing, we have for $t < t^*$, $u^*(t) = 1$, and, thus,

$$ {}_tD_T^\alpha p_1(t) = p_1(t) \tag{23}$$

The solution of this linear fractional differential equation (23) is given by $p(t) = p(t^*)\Phi_\alpha(t^*, t)$, where $p(t^*) = 1$ and $\Phi_\alpha(t^*, t)$ is the fractional state transition matrix (in fact, scalar-valued) that can be computed by the Mittag-Leffler function defined in the previous section. By setting $\beta = \alpha$, $A = [1]$ and by replacing t by $t^* - t = T - \Gamma(\alpha + 1)^{\alpha^{-1}} - t$, we conclude that

$$\begin{aligned} p_1(t) &= e_\alpha(1, t^* - t) \\ &= (t^* - t)^{\alpha - 1} E_{\alpha,\alpha}((t^* - t)^\alpha) \\ &= (t^* - t)^{\alpha - 1} \sum_{k=0}^{\infty} \frac{(t^* - t)^{k\alpha}}{\Gamma((k + 1)\alpha)}. \end{aligned}$$

Note that if $\alpha = 1$, then we have classical solution e^{T-t-1}.

Since we have the optimal control u^*, we can easily compute the optimal trajectory which satisfies $x^*(0) = x_0$, and

$$ {}_0^C D_t^\alpha x^*(t) = \begin{cases} x^*(t) & \text{if } t \in [0, t^*] \\ 0 & \text{if } t \in [t^*, T] \end{cases} $$

We can compute the optimal trajectory x^* by the generalization Mittag-Leffler function, $\forall t \in [0, t^*]$, $x^*(0) = x_0$, we conclude that

$$ \begin{aligned} x^*(t) &= E_\alpha(at^\alpha) \\ &= x_0 E_\alpha(t^\alpha) \\ &= x_0 \sum_{k=0}^{\infty} \frac{t^{k\alpha}}{\Gamma(k\alpha + 1)}. \end{aligned} $$

Note that if $\alpha = 1$, then we have classical solution $x_0 e^t$.

Now we compute the optimal trajectory x^* in the interval $[t^*, T]$, which $u^* = 0$, $x^*(t) = x^*(T)$, we conclude that

$$ \begin{aligned} x^*(t) &= x^*(t^*) \\ &= x_0 E_\alpha((t^*)^\alpha) \\ &= x_0 \sum_{k=0}^{\infty} \frac{(T - (\Gamma(\alpha + 1))^{\frac{1}{\alpha}})^{k\alpha}}{\Gamma(k\alpha + 1)}. \end{aligned} $$

Note that if $\alpha = 1$, then we have classical solution $x_0 e^{T-1}$.

5. Conclusion

This article concerns the derivation of necessary conditions of optimality in the form of Pontryagin maximum principle for a nonlinear fractional optimal control problem whose differential equation involves the Caputo derivative of the state variable with respect to time. Under mild assumptions on the data of the problem the proof involved the direct application of variational arguments, thus avoiding the often used argument of converting the optimal control problem into a conventional one and, then, express the optimality conditions for this auxiliary problem back in the fractional derivative context. Another interesting novelty consists in the fact that, unlike in most fractional optimal control problem formulations, we consider the cost functional given by a fractional integral of Riemann-Liouville type.

A simple example illustrating the application of our maximum principle was presented. The optimal control strategy was computed analytically being the fractional differential adjoint equation solved by using technique based on a generalization Mittag-Leffler function.

A natural sequel of this article concerns the weakening of the assumptions on the data of the problem. notably the mere measurability dependence of the dynamics with respect to time and to the control variables. This will certainly require more sophisticated variational arguments and the use of methods and results of nonsmooth analysis. Another direction of research consists in increasing the structure of the fractional optimal control problem by considering additional state endpoint constraints, and state and/or mixed constraints in its formulation. In this case, additional regularity assumptions will be needed to ensure that the obtained necessary conditions of optimality do not degenerate.

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